

SOLVABLE NORMAL SUBGROUPS OF 2-KNOT GROUPS

J.A.HILLMAN

ABSTRACT. If X is an orientable, strongly minimal PD_4 -complex and $\pi_1(X)$ has one end then it has no nontrivial locally-finite normal subgroup. Hence if π is a 2-knot group then (a) if π is virtually solvable then either π has two ends or $\pi \cong \Phi$, with presentation $\langle a, t | ta = a^2t \rangle$, or π is torsion-free and polycyclic of Hirsch length 4; (b) either π has two ends, or π has one end and the centre $\zeta\pi$ is torsion-free, or π has infinitely many ends and $\zeta\pi$ is finite; and (c) the Hirsch-Plotkin radical $\sqrt{\pi}$ is nilpotent.

The main result of this note (in §1) is that if X is an orientable PD_4 -complex such that $\pi_1(X)$ has one end and the equivariant intersection pairing on $\pi_2(X)$ is 0 then $\pi_1(X)$ has no nontrivial locally-finite normal subgroup. This has several applications to 2-knot groups and their subgroups.

In §2 we show that if a 2-knot group π is virtually solvable then either π' is finite or $\pi \cong \Phi = Z*_2$, with presentation $\langle a, t | ta = a^2t \rangle$ (the group of Fox's Example 10), or π is torsion-free polycyclic and of Hirsch length 4. Such groups are all known. (The final family was found in [6].) More generally, if S is an infinite solvable normal subgroup and π is not itself solvable then $S \cong \mathbb{Z}^2$ or is virtually torsion-free abelian of rank 1. We show also that the Hirsch-Plotkin radical $\sqrt{\pi}$ of every 2-knot group is nilpotent. Finally we consider the centre $\zeta\pi$. If π has one end then $\zeta\pi \cong \mathbb{Z}^2$ or is torsion-free, of rank ≤ 1 . In particular, this is so if the commutator subgroup π' is infinite and $\zeta\pi$ has rank > 0 . If π has two ends $\zeta\pi$ has rank 1, and may be either \mathbb{Z} or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, while if π has infinitely many ends $\zeta\pi$ is finite. We extend a construction of [12] to give examples with $\sqrt{\pi}$ cyclic of order q or $2q$, with q odd.

1. PD_4 -COMPLEXES WITH $\chi = 0$

A PD_4 -complex X with fundamental group π is *strongly minimal* if the equivariant intersection pairing on $\pi_2(X)$ is 0, equivalently, if

1991 *Mathematics Subject Classification.* 57Q45.

Key words and phrases. centre, coherent, Hirsch-Plotkin radical, 2-knot, torsion.

the homomorphism from $H^2(\pi; \mathbb{Z}[\pi])$ to $H^2(X; \mathbb{Z}[\pi])$ induced by the classifying map $c_X : X \rightarrow K(\pi, 1)$ is an isomorphism [5].

Lemma 1. *Let G be a group. If T is a locally finite normal subgroup of G then T acts trivially on $H^j(G; \mathbb{Z}[G])$, for all $j \geq 0$.*

Proof. If T is finite then $H^j(G; \mathbb{Z}[G]) \cong H^j(G/T; \mathbb{Z}[G/T])$, for all j , and the result is clear. Thus we may assume that T and G are infinite. Hence $H^0(G; \mathbb{Z}[G]) = 0$, and T acts trivially. We may write $T = \cup_{n \geq 1} T_n$ as a strictly increasing union of finite subgroups. Then there are short exact sequences [8]

$$0 \rightarrow \varprojlim^1 H^{s-1}(T_n; \mathbb{Z}[\pi]) \rightarrow H^s(T; \mathbb{Z}[\pi]) \rightarrow \varprojlim H^s(T_n; \mathbb{Z}[\pi]) \rightarrow 0.$$

Hence $H^s(T; \mathbb{Z}[\pi]) = 0$ if $s \neq 1$ and $H^1(T; \mathbb{Z}[\pi]) = \varprojlim^1 H^0(T_n; \mathbb{Z}[\pi])$, and so the LHS spectral sequence collapses to give $H^j(G; \mathbb{Z}[G]) \cong H^{j-1}(G/T; H^1(T; \mathbb{Z}[G]))$, for all $j \geq 1$. Let $g \in T$. We may assume that $g \in T_n$ for all n , and so g acts trivially on $H^0(T_n; \mathbb{Z}[G])$, for all j and n . But then g acts trivially on $\varprojlim^1 H^0(T_n; \mathbb{Z}[\pi])$, by the functoriality of the construction. Hence every element of T acts trivially on $H^{j-1}(G/T; H^1(T; \mathbb{Z}[G]))$, for all $j \geq 1$. \square

Theorem 2. *Let X be an orientable PD_4 -complex with fundamental group π . If X is strongly minimal and π has one end then π has no non-trivial locally-finite normal subgroup.*

Proof. Since π has one end, $H_s(X; \mathbb{Z}[\pi]) = 0$ for $s \neq 0$ or 2 . Poincaré duality and c_X give an isomorphism $\Pi = H_2(X; \mathbb{Z}[\pi]) \cong H^2(X; \mathbb{Z}[\pi])$. Since X is strongly minimal, this in turn is isomorphic to $H^2(\pi; \mathbb{Z}[\pi])$.

Suppose that π has a nontrivial locally-finite normal subgroup T . Let $g \in T$ have prime order p , and let $C = \langle g \rangle \cong Z/pZ$. We apply Lemma 2.10 of [4], to conclude that $H_{i+3}(C; \mathbb{Z}) \cong H_i(C; \Pi)$ for all $i \geq 2$. Since C has cohomological period 2 and acts trivially on Π , by Lemma 1, there is an exact sequence

$$0 \rightarrow Z/pZ \rightarrow \Pi \rightarrow \Pi \rightarrow 0.$$

But $\Pi \cong H^2(\pi; \mathbb{Z}[\pi])$ is torsion-free, by Proposition 13.7.1 of [2], since π is finitely presentable. Hence T has no such element g and so π has no such finite normal subgroup. \square

Corollary 2.1. *Every locally-finite ascending subgroup of π is trivial.*

Proof. If T is a locally-finite ascending subgroup of π then a transfinite induction shows that the normal closure of T in π is locally finite. \square

Let $\beta_i^{(2)}(X)$ be the i th L^2 Betti number of X . (See [9] for a comprehensive exposition of L^2 -theory.)

Lemma 3. *Let X be a finite PD_4 -complex with fundamental group π . If $\chi(X) = 0$ and $\beta_1^{(2)}(\pi) = 0$ then X is strongly minimal.*

Proof. Since X is a finite complex the L^2 -Euler characteristic formula holds, and so $\chi(X) = \beta_2^{(2)}(X) - 2\beta_1^{(2)}(X)$. Hence $\beta_2^{(2)}(X) = 0$ also. Since $\beta_1^{(2)}(X) = \beta_1^{(2)}(\pi) = 0$, $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi) \geq 0$ and $\chi(X) = 0$, it follows that $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$. Hence $H^2(c_X; \mathbb{Z}[\pi])$ is an isomorphism, by part (3) of Theorem 3.4 of [4]. \square

We shall apply these results to 2-knot groups in the next section.

2. CENTRES, HIRSCH-PLOTKIN RADICALS AND VIRTUALLY SOLVABLE 2-KNOT GROUPS

Let K be a 2-knot with exterior $X(K) = S^4 \setminus K$ and group $\pi = \pi K = \pi_1(X(K))$, and let $M(K) = X(K) \cup S^1 \times D^3$ be the closed 4-manifold obtained by elementary surgery on K in S^4 . Then $\pi_1(M(K)) \cong \pi$ and $\chi(M(K)) = 0$.

Let $\zeta G, G'$ and \sqrt{G} denote the centre, commutator subgroup and Hirsch-Plotkin radical of a group G , respectively. If G is elementary amenable then it has a well-defined Hirsch length $h(G) \in \mathbb{N} \cup \{\infty\}$. (See [7] or Chapter 1 of [4].)

Theorem 4. *Let K be a 2-knot with group $\pi = \pi K$. If π has normal subgroups $A \leq E$ with A a nontrivial abelian group and E an infinite elementary amenable group then either π' is finite or E is virtually torsion-free solvable. If $h(E) = 1$ then E is abelian or virtually \mathbb{Z} ; if $h(E) = 2$ then $E \cong \mathbb{Z}^2$, and if $h(E) > 2$ then E is torsion-free polycyclic, and $h(E) = 3$ or 4 .*

Proof. We may assume that π' is infinite. Then π has one end, and $\beta_1^{(2)}(\pi) = 0$, since π has an infinite elementary amenable normal subgroup. Since $M(K)$ is a closed 4-manifold, it is homotopy equivalent to a finite PD_4 -complex, and so is strongly minimal, by Lemma 3. The torsion subgroup of A is characteristic, and so is normal in π . Hence A is torsion-free, by Theorem 2. Therefore either $\pi \cong \Phi$ or $M(K)$ is aspherical or $A \cong \mathbb{Z}$ and π/A has infinitely many ends, by Theorem 15.8 of [4]. In all cases E is in fact virtually torsion-free solvable. (This is Corollary 1.9.2 of [4] when $M(K)$ is aspherical.)

If E is infinite then π has one or two ends. Hence if $h(E) = 1$ and π' is infinite then E has no finite normal subgroup, and so must have a torsion-free abelian subgroup of index ≤ 2 . If E is not finitely generated then $\pi \cong \Phi$ or $M(K)$ is aspherical, by Theorem 15.8 of [4], and so π is torsion-free. Hence S must be abelian.

If $h(E) = 2$ then $M(K)$ is aspherical and E is torsion-free and virtually \mathbb{Z}^2 , by Theorems 9.1 and 16.2 of [4]. Since $M(K)$ is orientable, E cannot be the Klein bottle group, and so $E \cong \mathbb{Z}^2$. If $h(E) > 2$ then π is torsion-free polycyclic and $h(\pi) = 4$, by Theorem 8.1 of [4], so $h(E) = 3$ or 4 . \square

We do not know whether E must be abelian when π' is infinite, E is finitely generated and $h(E) = 1$ or 2 .

Corollary 4.1. *If π is virtually solvable then either π' is finite or $\pi \cong \Phi$ or π is torsion-free polycyclic and $h(\pi) = 4$.*

Proof. If π is virtually solvable it has a solvable normal subgroup S of finite index. The lowest nontrivial term of the derived series for S is characteristic in S , and so normal in π . Hence the theorem applies. \square

It is enough to assume that π is elementary amenable and has a nontrivial abelian normal subgroup. Can we relax “virtually solvable” further to just “elementary amenable”?

If $\pi \cong \Phi$ then K is TOP isotopic to Fox’s Example 10 (or its reflection) [5], while if π is torsion-free polycyclic then it determines $M(K)$ is determined up to homeomorphism, by Theorem 17.4 of [4].

The product of locally-nilpotent normal subgroups of a group G is again a locally-nilpotent normal subgroup, by the Hirsch-Plotkin Theorem, and the Hirsch-Plotkin radical \sqrt{G} is the (unique) maximal such subgroup. (See Proposition 12.1.2 of [10].) This subgroup contains all the nilpotent normal subgroups of G , and is clearly nilpotent if it is finitely generated. However in general \sqrt{G} need not be nilpotent.

Corollary 4.2. *The Hirsch-Plotkin radical $\sqrt{\pi}$ is nilpotent.*

Proof. Since $\sqrt{\pi}$ is locally nilpotent it has a maximal locally-finite normal subgroup T with torsion-free quotient. If $\sqrt{\pi}$ is finitely generated there is nothing to prove. Otherwise, π has one end and so T is trivial, by Theorem 3. If $T = 1$ and $h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi}$ is abelian; if $h(\sqrt{\pi}) > 2$ then π is virtually polycyclic and $h(\pi) = 4$, by Theorem 15.8 of [4]. In each case $\sqrt{\pi}$ is nilpotent. \square

Every finitely generated abelian group is the centre of some high-dimensional knot group [3]. On the other hand, the only classical knots whose groups have nontrivial abelian normal subgroups are the torus knots, for which $\sqrt{\pi} = \zeta\pi \cong \mathbb{Z}$ and $\zeta\pi \cap \pi' = 1$. The intermediate case of 2-knots is less clear. If $\zeta\pi$ has rank > 1 then it is \mathbb{Z}^2 ; most twist spins of torus knots have such groups. There are examples with centre 1 , $Z/2Z$, \mathbb{Z} or $\mathbb{Z} \oplus Z/2Z$. (See Chapters 15–17 of [4].)

Corollary 4.3. *Let K be a 2-knot with group $\pi = \pi K$. Then*

- (1) *if π has two ends then $\zeta\pi \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if π' has even order; otherwise $\zeta\pi \cong \mathbb{Z}$;*
- (2) *if π has one end then $\zeta\pi \cong \mathbb{Z}^2$, or is torsion-free of rank ≤ 1 ;*
- (3) *if π has infinitely many ends then $\zeta\pi$ is finite.*

Proof. If π has two ends then π' is finite, and so $\zeta\pi$ is finitely generated and of rank 1. It follows from the classification of such 2-knot groups (see §4 of Chapter 15 of [4]) that $\zeta\pi \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if π' has even order, and otherwise $\zeta\pi \cong \mathbb{Z}$.

Part (2) follows from Theorem 4, while part (3) is clear. \square

Note that π has finitely many ends if π' is finitely generated, or if $\zeta\pi$ is infinite. When π has more than one end Lemma 2.10 of [4] either does not lead to a contradiction or does not apply.

If $\zeta\pi$ is a nontrivial torsion group then it is finite. Yoshikawa constructed an example of a 2-knot whose group π has centre of order 2 [13]. It is easy to see that $\sqrt{\pi} = \zeta\pi$ in this case. The construction may be extended as follows. Let $q > 0$ be odd and let k_q be a 2-bridge knot such that the 2-fold branched cyclic cover of S^3 , branched over k_q is a lens space $L(3q, r)$, for some r relatively prime to q . Let $K_1 = \tau_2 k_q$ be the 2-twist spin of k_q , and let $K_2 = \tau_3 k$ be the 3-twist spin of a nontrivial knot k . Let γ be a simple closed curve in $X(K_1)$ with image $[\gamma] \in \pi K_1$ of order $3q$, and let w be a meridian for K_2 . Then w^3 is central in πK_2 . The group of the satellite of K_1 about K_2 relative to γ is the generalized free product

$$\pi = \pi K_2 / \langle \langle w^{3q} \rangle \rangle *_{w=[\gamma]} \pi K_1.$$

(see §14.3 of [4].) Hence $\sqrt{\pi} = \langle w^3 \rangle \cong \mathbb{Z}/q\mathbb{Z}$, while $\zeta\pi = 1$.

If we use a 2-knot K_1 with group $(Q(8) \times \mathbb{Z}/3q\mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ instead and choose γ so that $[\gamma]$ has order $6q$ then we obtain examples with $\sqrt{\pi} \cong \mathbb{Z}/2q\mathbb{Z}$ and $\zeta\pi = \mathbb{Z}/2\mathbb{Z}$. (Knots K_1 with such groups may be constructed by surgery on sections of mapping tori of homeomorphisms of 3-manifolds with fundamental group $Q(8) \times \mathbb{Z}/3q\mathbb{Z}$ [12].)

If $\zeta\pi$ has rank 1 and nontrivial torsion then π' is finite, and $\zeta\pi$ is finitely generated.

If $\zeta\pi$ has rank 1 but is not finitely generated then $M(K)$ is aspherical. It is not known whether there are such 2-knots (nor, more generally, whether abelian normal subgroups of rank 1 in PD_n -groups with $n > 3$ must be finitely generated). What little we know about this case is as follows. Since $\zeta\pi < \pi'$ and $\pi/\pi' \cong \mathbb{Z}$, we must have $\zeta\pi \leq \pi''$. Since $\zeta\pi$ is torsion-free of rank 1 but is not finitely generated, *c.d.* $\zeta\pi = 2$. Hence if G is a nonabelian subgroup which contains $\zeta\pi$ then *c.d.* $G \geq 3$, by

Theorem 8.6 of [1]. If H is a subgroup of π such that $H \cap \zeta\pi = 1$ then $H.\zeta\pi \cong H \times \zeta\pi$ is not finitely generated, and so has infinite index in π . Hence $c.d.H \times \mathbb{Z} \leq c.d.H \times \zeta\pi \leq 3$ [11]. Theorem 5.5 of [1] gives, firstly, that $c.d.H \leq 2$, and then, that if H is FP_2 then $c.d.H \leq 1$, and so H is free. Thus if π is almost coherent every subgroup either meets $\zeta\pi$ nontrivially or is locally free.

If $\zeta\pi$ has rank > 1 then $M(K)$ is aspherical and $\zeta\pi \cong \mathbb{Z}^2$, by Theorem 16.3 of [4].

The following questions remain open:

- (i) if $\zeta\pi$ has rank 1, must it be finitely generated?
- (ii) if $\zeta\pi$ is finite, must it be $\mathbb{Z}/2\mathbb{Z}$ or 1?
- (iii) is there a 2-knot group π with $\sqrt{\pi}$ a non-cyclic finite group?
- (iv) if π is elementary amenable is it virtually solvable?

In each case the answer is “yes” if π' is finitely presentable, for then the infinite cyclic cover $M(K)'$ is homotopy equivalent to a PD_3 -complex, by Theorem 4.5 of [4].

At the time of writing, the largest known class of groups for which 4-dimensional TOP surgery works is the class SA obtained from subexponential groups by taking increasing unions and extensions. Are there 2-knot groups in this class which are not virtually solvable?

REFERENCES

- [1] Bieri, R. *Homological Dimension of Discrete Groups*, Queen Mary College Lecture Notes in Mathematics, London (1976).
- [2] Geoghegan, R. *Topological Methods in Group Theory*, Graduate Texts in Mathematics 243, Springer-Verlag, Berlin – Heidelberg – New York (2008).
- [3] Hausmann, J.-C. and Kervaire, M. Sur le centre des groupes de noeuds multidimensionnelles, C.R. Acad. Sci. Paris 287 (1978), 699–702.
- [4] Hillman, J. A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs 5, Geometry and Topology Publications (2002). (Revisions 2007 and 2014).
- [5] Hillman, J.A. Strongly minimal PD_4 -complexes, Top. Appl. 156 (2009), 1565–1577.
- [6] Hillman, J.A. and Howie, J. Seifert fibred knot manifolds, J. Knot Theory Ramif. 22 (2013), 1350082.
- [7] Hillman, J.A. and Linnell, P.A. Elementary amenable groups of finite Hirsch length are locally finite by virtually solvable, J. Austral. Math. Soc. 52 (1992), 237–241.
- [8] Jensen, C.U. *Les foncteurs dérivés de \varprojlim et ses applications a la théorie des modules*, Lecture Notes in Mathematics 254, Springer-Verlag, Berlin – Heidelberg – New York (1972).
- [9] Lück, W. *L^2 -Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Bd. 44, Springer-Verlag, Berlin - Heidelberg - New York (2002).
- [10] Robinson, D.J.S. *A Course in the Theory of Groups*, Graduate Texts in Mathematics 80, Springer-Verlag, Berlin – Heidelberg – New York (1982).
- [11] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups, Comment. Math. Helvetici 52 (1977), 317–324.
- [12] Yoshikawa, K. On 2-knot groups with the finite commutator subgroups, Math. Seminar Notes Kobe University 8 (1980), 321–330.
- [13] Yoshikawa, K. On a 2-knot group with nontrivial centre, Bull. Austral.Math. Soc. 25 (1982), 321–326.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, NSW
2006, AUSTRALIA

E-mail address: jonathan.hillman@sydney.edu.au